Lifting of L-narrowing derivations

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Abstract

If conditional rewrite-rules are restricted to the form $P \Longrightarrow f(x_1, \ldots, x_n) \rightarrow t$ where P is a finite set of equations, f is any function symbol, $x_1, \ldots, x_n$ are variables, and t is any term then the premise P contains in general variables which do not occur in the list $x_1, \ldots, x_n$. The rule with premise P can be applied if P is satisfiable. Therefore, we need methods to solve P and narrowing must be combined with rewriting. But, narrowing becomes a special case, called L-narrowing, closely related to lazy-narrowing. Two lifting lemmas are shown which characterize the relationship of L-narrowing derivations if the goals are modified by substitutions. From these lifting lemmas, soundness and completeness results can be concluded.

1 Introduction

In order to prove soundness and completeness of narrowing the relationship between rewriting and narrowing is used. So, it was done for instance in [Hul80],[Pa88]. This method is based on the fact that rewriting is sound and under certain conditions also complete for proving equations in varieties of term algebras. In the case of conditional rules of the form $P \Longrightarrow s \rightarrow t$ where P represents a finite set of equations, s and t are terms, the proplem becomes more complicated. For rewriting, these rules are often restricted in such a way that all variables occurring in P or t have to occur in s too. So, for canonical systems completeness was proved in [Huss86]. In this paper, we consider a special form of conditional rules $P \Longrightarrow f(x_1, \ldots, x_n) \rightarrow t$ where the term s consists only of the function symbol f and the variables $x_1, \ldots, x_n$. It it easy to see that each rule

$$P \Longrightarrow f(t_1, \ldots, t_n) \rightarrow t$$

can be transformed into an 'equivalent' rule

$$P \cup \{x_1 = t_1, \ldots, x_n = t_n\} \Longrightarrow f(x_1, \ldots, x_n) \rightarrow t.$$

That means: we do not lose any expressive power by this restriction. We consider this kind of rules because of their simple form which induces some special properties.

Firstly, the match of such a rule with a subterm s/u is very simple. One has only to check whether the function symbol f occurs at u in s. The variable x_i then always matches with s/ui. That means, the match is always replaced by the more general unification. Therefore, confluence is more often fulfilled than in the general case. To demonstrate this we take the following example. Let $+$ be the usual addition of natural numbers and h an unary function symbol denoting the partially defined integer division by 2. The function h can be defined without recursion by:

$$h(x+x) \rightarrow x.$$

The term $h(0)$ is irreducible since 0 does not match with $x + x$. In order to obtain a canonical system we had to apply a completion algorithm. However, let us transform this rule into the restricted form

$$\{y = x + x\} \Longrightarrow h(y) \rightarrow x.$$

This rule can be applied onto $h(0)$ since y matches with 0 and we get

$$\{0 = x + x\} \Longrightarrow h(0) \rightarrow x.$$

Secondly, our restricted rules are left-linear (all variables occur in $f(x_1, \ldots, x_n)$ at most once). As a consequence, we have more outermost redices.

Finally, using our kind of rules, narrowing finds more solution than in the general case. For instance, for the rules

$$a \rightarrow b\ ,\ f(a,b) \rightarrow b$$

narrowing fails to find a solution for the goal

$$f(x,x) = a.$$

But, both the solutions $x = a$ and $x = b$ can be found by narrowing using the rules

$$a \rightarrow b\ ,\ \{x_1 = a, x_2 = b\} \Longrightarrow f(x_1,x_2) \rightarrow b.$$

Note, that $x = a$ is even a reducible solution.

These properties stimulated us to investigate these restricted rules in more details. Our aim is to use them as operational semantics for a logic and functional language like it is done in [GHR93] with some other kind of rules.

However, there is one consequence using our special kind of rules: in general, there will be variables occurring in the premise which do not occur in $f(x_1, \ldots, x_n)$. We call these variables *auxillary variables* since they play an auxillary role: the solutions for the auxillary variables are not of any interest wrt. the solutions of the original goal. In the usual approaches, an equation system (goal) is called valid (it holds) if it is fulfilled for all instantiations of its variables. That means, it holds if all variables included in the goal have the identical substitution as solution. Unfortunately, as we will show in chapter 3, due to the auxillary variables this concept must be slightly generalized.

A rule can be applied if the corresponding premise is fulfilled, i.e. if it has a solution. Now, we need narrowing within rewriting in order to find solutions of the premise. Rewriting and narrowing must be combined.

In order to solve a goal S we allow two kind of steps:

If there is a most general unifier μ for one equation of the goal then μ is a part of the solution. In this case the corresponding equation can be removed from S and the new goal consists of the remaining equations modified by μ. This is called a *reflecting step*.

If no equation of S can be unified then we try to narrow some equation by a *narrowing step* to bring it closer to a reflecting step. We apply one of our rule $P \Longrightarrow f(x_1, \ldots, x_n) \rightarrow t$ onto a subterm of the goal S. But, this is only allowed if all equations from P are fulfilled. We postpone this check by adding P to S where both of them are modified by a substitution induced by the match. This union becomes the new goal. A sequence of these steps is called a derivation.

This method is related to lazy-narrowing described in [Pa88]. Namely, if the root (the head symbol) of a rule coincides with a term then this rule will be applied. Instead the match new equations, which represent this match, are added to the goal. So, the match is replaced by unification. This is also done by our rules. However, we do not demand an outermost strategy. Narrowing with our kind of rules we will call *L-narrowing*.

Our main aim was to investigate the behaviour of L-narrowing with respect to modifications of the goal caused by substitutions. There are two questions in this connection.
Question 1 (*Forward lifting*): If we know a derivation starting with goal S and δ is a certain substitution then we ask whether there is an analogous derivation, i.e. with the same steps in the same order, starting with $S\delta$ (the application of δ on S) and what is the result?
Question 2 (*Backward lifting*): If we know a derivation starting with $S\delta$ then we ask whether there is a derivation starting with S and what is the result?

Lifting lemmas are already known for resolution, paramodulation, and narrowing ([GH90], [Pa88]). However, they describe a different situation. In [Pa88], for instance, the relationship between a narrowing expansion, and a *most general* narrowing expansion, where always the most general unifier is used, is considered. L-narrowing uses only the most general unifier. We are especially interested in the characterization of the resulting substitutions influenced by lifting if the same steps in the same order are applied. For forward lifting, this can be done by means of *weakest unifiers*. From the lifting lemmas presented in this paper we can conclude soundness and completeness of L-narrowing which is, of course, already known.

In section 1 we introduce our used notations and prove some basic properties. Since forward lifting can be characterized by weakest unifiers, this notation plays a special role. We apologize for using own notations in some cases. The most notations are standard. But, in order to shorten the presentation some new notations were introduced in section 2. In section 3, L-narrowing derivations are defined. Also here, some propositions describe the basic behaviour. The sections 4 and 5 contain the lifting lemmas as well as soundness and completeness results.

2 Notations and basic properties

Let $\mathbb{N}$ be the set of natural numbers.

For any set A the set of all sequences of elements of A (words over A) of length n is denoted by A^n, A^* denotes the set of all words over A, and ϵ is the empty word ($\epsilon \in A^0$). If $\underline{a} \in A^n$ then a_i is the i-th element of $\underline{a}$, i.e. $\underline{a} = a_1 \dots a_n$.

By F we denote a ranked alphabet, also called *signature*, which is provided with an arity-mapping $ary : F \to \mathbb{N}$. If $f \in F$ and $ary(f) = n$ then f denotes an n-ary function symbol. Sometimes we write $f \in F_n$ instead of $f \in F$ and $ary(f) = n$. Note, that here only the homogeneous case is considered, i.e. there is only one sort of function symbols.

By means of F we build terms in the usual way. The set $\mathcal{T}(F)$ of these terms is the least set for which the implication $f \in F_n \wedge \underline{t} \in \mathcal{T}(F)^n \implies \underline{t}f \in \mathcal{T}(F)$ holds.

As variables we use the set X which is the disjoint union of the three sets X_s, $X_{\mathbb{N}}$, and $X^{()}$. This partition was done by technical reasons. Here, X_s denotes the set of *standard* variables, $X_{I\!N} := \{x_n \ : \ n \in I\!N\}$ is a special set of *positional* variables, and the elements of the special set $X^{()} := \{x^{(n)} \ : \ n \in I\!N\}$ are called *auxillary* variables. As an abbreviation we introduce $X_n := \{x_1, \ldots, x_n\} \subset X_{I\!N}$.

All kinds of variables are introduced and used in a special way by rules and narrowing derivations. All variables have the arity zero, i.e. if $x \in X$ then $ary(x) = 0$.
If $Y \subseteq X$ then $\mathcal{T}(F \cup Y)$ is the set of all terms which may include variables from Y. So, if $x_1, x_2, \ldots, x_n \in X_n$ are positional variables and $f \in F_n$ then $x_1 x_2 \ldots x_n f \in \mathcal{T}(F \cup X_s)$ is a term containing the symbol f and positional variables. Since $\underline{x} = x_1 x_2 \ldots x_n$, this term can be written as $\underline{x}f$.

Additionally to the elements of the signature F we use two special symbols in order to build terms: $\asymp$ and Λ. We set $ary(\asymp) := 2$, but $ary(\Lambda)$ is undefined, that means Λ can have any arity. A term $st\asymp$ where $s, t \in \mathcal{T}(F \cup Y)$, $Y \subseteq X$ is called an *equation*. $EQ(F \cup Y) := \{st\asymp \ : \ s, t \in \mathcal{T}(F \cup Y)\}$ denotes the set of all equations.

By means of Λ we can build for any $n \in I\!N$ and $\underline{e} \in EQ(F \cup Y)^n$ a term $\underline{e}\Lambda$, called an *equation system of length n*. $ES(F \cup Y)$ denotes the set of all equation systems of any length. Another type of terms cannot be built using $\asymp$ and Λ. Note, that Λ itself is also a term, namely an equation system of length 0 called the *empty system*.

Finally, we set $\hat{\mathcal{T}} := \hat{\mathcal{T}}(F \cup X) := \mathcal{T}(F \cup X) \cup EQ(F \cup X) \cup ES(F \cup X)$ and

$$F_X := F \cup X \cup \{\asymp, \Lambda\} \text{ where } F_{X,n} := \begin{cases} F_0 \cup X \cup \{\Lambda\} & \text{if } n = 0, \\ F_2 \cup \{\asymp, \Lambda\} & \text{if } n = 2, \\ F_n \cup \{\Lambda\} & \text{otherwise.} \end{cases}$$

Positions in a term can uniquely be described by *occurrences*, a finite subset of $I\!N_+^*$, where $I\!N_+ := I\!N \backslash \{0\}$. For a term t the set $occ(t)$ of all occurrences is inductively defined by

$$\text{If } f \in F_{X,n} \text{ and } \underline{t} \in \hat{\mathcal{T}}^n \text{ then } occ(\underline{t}f) := \{\epsilon\} \cup \{i\alpha \ : \ 1 \le i \le n \text{ and } \alpha \in occ(t_i)\}.$$

If $t \in \hat{\mathcal{T}}$ and $u \in occ(t)$ then $t(u) \in F_X$ means the symbol at occurrence u in t and t/u means the subterm of t at u, defined by:

$$\underline{t}f(\epsilon) := f, \quad \underline{t}f/\epsilon := \underline{t}f, \text{ and}$$
$$\text{if } f \in F_{X,n}, \underline{t} \in \hat{\mathcal{T}}^n, \text{ and } 1 \le i \le n \text{ then } \quad \underline{t}f(iu) := t_i(u), \ \ \underline{t}f/iu := t_i/u.$$

If in the term t the subterm t/u is replaced by the term s then the term $t[u \leftarrow s]$ is the result, formally defined by:

$$t[\epsilon \leftarrow s] := s \text{ and}$$
$$\text{if } f \in F_{X,n}, \underline{t} \in \hat{\mathcal{T}}^n \text{ and } 1 \le i \le n \text{ then } \quad \underline{t}f[iu \leftarrow s] := t_1 \ldots t_i[u \leftarrow s] \ldots t_n f.$$

If S is an equation system of length n and $1 \le i \le n$ then $S[i \leftarrow \epsilon]$ means the replacement of the equation i by the emtpy word, i.e. the deletion of equation i. This can be formally defined by: $e_1 \ldots e_n \Lambda[i \leftarrow \epsilon] := e_1 \ldots e_{i-1} e_{i+1} \ldots e_n \Lambda$.

Moreover, for equation systems we introduce a special *join* operation $\bowtie$, defined by

$$\underline{e}\Lambda \bowtie \underline{e}'\Lambda := \underline{e}\underline{e}'\Lambda.$$

Sometimes we are especially interested in those positions of a term where function symbols (or variables respectively) occur. These occurrences are denoted by $focc(t)$ and $vocc(t)$:

$$focc(t) := \{u \ : \ u \in occ(t) \wedge t(u) \in F\}, \ vocc(t) := \{u \ : \ u \in occ(t) \wedge t(u) \in X\}.$$

The set of all variables occurring in t can now be defined as:

$$var(t) := \{t(u) \ : \ u \in vocc(t)\}.$$

A *substitution* is a function of type $\varphi : X \to \mathcal{T}(F \cup X)$. For substitutions we write $x\varphi$ instead of $\varphi(x)$ in order to describe the result of the application of φ onto x. Each substitution has a unique extension $\varphi^* : \hat{\mathcal{T}} \to \hat{\mathcal{T}}$ by:

$$\text{if } x \in X \text{ then } x\varphi^* := x\varphi, \ \text{ if } f \in F_{X,n} \text{ and } \underline{t} \in \hat{\mathcal{T}}^n \text{ then } \underline{t}f\varphi^* := t_1\varphi^* \dots t_n\varphi^* f.$$

In general, we do not distinguish between φ and φ^* and omit the $*$.

The *domain* of a substitution φ: $dom\varphi := \{x \ : \ x\varphi \neq x\}$ is the set of all variables which are changed by φ, the $range\varphi := \bigcup\{var(x\varphi) \ : \ x \in dom\varphi\}$ is the set of all variables occurring in images. We set $\overline{dom\varphi} := X \backslash dom\varphi$.

The restriction $\varphi \downarrow Y$ of φ on $Y \subseteq X$ is a substitution with $dom(\varphi \downarrow Y) \subseteq Y$ and $\forall x \in Y : x(\varphi \downarrow Y) = x\varphi$.

By λ we denote the substitution with empty domain, i.e. for all variables$x$: $x\lambda = x$ holds. So, for any substitution φ we have: $\lambda = \varphi \downarrow \emptyset$.

In this paper only *finite* substitutions, i.e. substitutions with finite domains are considered.

Substitutions can be composed: $\varphi\psi$ is the substitution with $t(\varphi\psi) = (t\varphi)\psi$. Obviously, we have $\varphi\lambda = \lambda\varphi = \varphi$ for all substitutions φ. We have always

$$range\varphi\psi \subseteq (range\varphi \cup range\psi) \text{ and } dom\varphi\psi \subseteq (dom\varphi \cup dom\psi).$$

The composition of substitutions induces a preorder $\sqsubseteq$: $\varphi \sqsubseteq \psi$ iff $\exists\gamma : \ \varphi\gamma = \psi$ and $\sim := \sqsubseteq \cap \sqsubseteq^{-1}$ is an equivalence relation where $\varphi \sim \psi$ iff $\varphi \sqsubseteq \psi$ and $\psi \sqsubseteq \varphi$.

It is well known that $\varphi \sim \psi$ iff $\exists$*renaming* $\rho : \ \varphi\rho = \psi$, where a renaming is a bijective substitution $\rho : X \to X$. So, for each renaming ρ the inverse ρ^{-1} is a renaming too and $\rho\rho^{-1} = \rho^{-1}\rho = \lambda$.

An *unifier* of a pair (s, t) of terms is a substitution φ such that $s\varphi = t\varphi$. An unifier μ of (s, t) is called a *most general unifier* (abbreviated: $\mu = mgu(s, t)$) if for every unifier φ of (s, t) the relation $\mu \sqsubseteq \varphi$ holds. If an unifier of (s, t) exists then the $mgu(s, t)$ exists too and it is uniquely determined up to a renaming.

If e is an equation then it has the form $st\asymp$ and $s = e/1$, $t = e/2$. Therefore, we set $mgu(e) := mgu(e/1, e/2)$, in case it exists.

If δ and σ are substitutions then we call a substitution φ an unifier of (δ, σ) if $\delta\varphi = \sigma\varphi$. Similarly, we call μ a most general unifier of (δ, σ), abbreviated by $\mu = mgu(\delta, \sigma)$, if for every unifier ψ of (δ, σ) the relation $\mu \sqsubseteq \psi$ holds.

In [GH90] it is shown that if for finite sets of pairs of terms an unifier exists then a most general unifier μ also exists. So, for finite substitutions we have the following lemma:

Lemma 2.1 *If for (δ, σ) an unifier exists then $mgu(\delta, \sigma)$ exists too.*

Proof: Let φ be any unifier of (δ, σ). Since the substitutions δ, σ are finite the set $P := \{(x\delta, x\sigma) \ : \ x \in dom\delta \cup dom\sigma\}$ is finite too and φ is an unifier for this set. Therefore, a mgu μ exists for this set and $\mu\psi = \varphi$ holds for a suitable ψ.
If $x \in dom\delta \cup dom\sigma$ then $x\delta\mu = x\sigma\mu$ because of $(x\delta, x\sigma) \in P$. If $x \notin dom\delta \cup dom\sigma$ then $x\delta\mu = x\mu = x\sigma\mu$. Together we get $\delta\mu = \sigma\mu$ and therefore, $\mu = mgu(\delta, \sigma)$. □

A substitution φ is called *idempotent* if $\varphi\varphi = \varphi$. A substitution φ is idempotent iff it has disjoint domain and range, i.e. $dom\varphi \cap range\varphi = \emptyset$. It is known [GH90] that if a most general unifier of a finite set of term-pairs exists then an idempotent mgu exists too. Therefore, we only accept idempotent most general unifiers. In accordance to the proof of lemma 2.1 a mgu of two substitutions is also idempotent! Additional, μ can be chosen such that $dom\mu$ and $range\mu$ contain only variables occurring in the unified terms. This reflects to the most general unifier of two substitutions. We introduce:

$$VAR(\varphi, \psi) \ := \ \{x \ : \ \exists y \in dom\varphi \cup dom\psi \ : \ x \in var(y\varphi) \cup var(y\psi)\}$$

Obviously, we have $range\varphi \cup range\psi \subseteq VAR(\varphi, \psi)$ and therefore, we can choose $\mu = mgu(\varphi, \psi)$ such that $dom\mu \subseteq VAR(\varphi, \psi)$.

Two substitutions δ, σ are called *consistent* if they have a common upper bound wrt. preorder $\sqsubseteq$, i.e. there are substitutions δ', σ' such that $\delta\delta' = \sigma\sigma'$. The pair (δ', σ') is also called a *weak unifier* of (δ, σ).

Definition 2.2 *A pair (δ', σ') is called a reduced weak unifier of (δ, σ) if it is a weak unifier and* $dom\delta \cap dom\delta' = dom\sigma \cap dom\sigma' = \emptyset$ *holds.*
(δ', σ') is called a weakest unifier of (δ, σ) (abbreviated: $(\delta', \sigma') = wu(\delta, \sigma)$) if additionally $dom\delta' \cup dom\sigma' \subseteq VAR(\delta, \sigma)$ and $\delta\delta' = \sigma\sigma' \sqsubseteq \varphi$ holds for all upper bounds φ of (δ, σ).

In order to get an analogous result to lemma 2.1 we consider the restricted case of idempotent substitutions.

Let δ and σ be two idempotent and consistent substitutions. It is easy to see that then a reduced weak unifier (δ', σ') exists.

We call two substitutions δ and σ *compatible* iff

$$\forall x (x \in \ dom\delta \cap dom\sigma \ \Rightarrow \ x\delta = x\sigma)$$

For the following propositions let (δ', σ') be a reduced weak unifier of the idempotent and consistent substitutions δ, σ.

Proposition 2.3 *δ' and σ' are compatible.*

Proof: We take any $x \in \ dom\delta' \cap dom\sigma'$. Therefore, by our precondition: $x \notin dom\delta$ and $x \notin dom\sigma$. It follows $x\delta' = x\delta\delta' = x\sigma\sigma' = x\sigma'$. □

Proposition 2.4 $\forall x (x \notin dom\delta \wedge x \in dom\sigma' \ \Rightarrow \ x \in dom\delta')$.

Proof: From $x \in dom\sigma'$, it follows: $x \notin dom\sigma$ and therefore,

$$x\delta' = x\delta\delta' = x\sigma\sigma' = x\sigma'.$$ □

For compatible substitutions δ and σ we define the operation *sum* $(\delta + \sigma)$ by:

$$x(\delta + \sigma) \ = \ \begin{cases} x\delta & \text{if} \quad x \in dom\delta \\ x\sigma & \text{if} \quad x \in dom\sigma \end{cases}$$

Proposition 2.5 $\delta(\delta' + \sigma') = \delta\delta'$ *and* $\sigma(\delta' + \sigma') = \sigma\sigma'$.

Proof: We prove only the first part, the second one follows symmetrically.
<u>Case 1:</u> $x \in dom\delta$. For any $y \in range\delta$ we have $y \notin dom\delta$.
<u>Case 1.1:</u> $y \in dom\sigma'$. By proposition 2.4 we have $y \in dom\delta'$ and therefore, by proposition 2.3 $y(\delta' + \sigma') = y\delta'$ follows.
<u>Case 1.2:</u> $y \notin dom\sigma'$. It follows immediately that $y(\delta' + \sigma') = y\delta'$.

Therefore, in case 1 we have for x : $x\delta(\delta' + \sigma') = x\delta\delta'$.
Case 2: $x \notin dom\delta$.
Case 2.1: $x \in dom\sigma'$. Therefore, by propositions 2.4 and 2.3: $x \in dom\delta'$ and $x\sigma' = x\delta'$, i.e. $x\delta(\delta' + \sigma') = x\delta\delta'$.
Case 2.2: $x \notin dom\sigma'$. That means $x\delta(\delta' + \sigma') = x\delta\delta'$. □

Lemma 2.6 *If two idempotent substitutions are consistent then they have a weakest unifier.*

Proof: Let δ and σ be two idempotent substitutions and let (δ', σ') be a reduced weak unifier.
By proposition 2.5 we have: $\delta(\delta' + \sigma') = \delta\delta' = \sigma\sigma' = \sigma(\delta' + \sigma')$. Therefore, $(\delta' + \sigma')$ is a unifier of (δ, σ). $\mu = mgu(\delta, \sigma)$ exists and there is a substitution ψ with $\mu\psi = \delta' + \sigma'$.
We build the substitutions $\mu_1 := \mu\!\downarrow\!(\overline{dom\delta})$ and $\mu_2 := \mu\!\downarrow\!(\overline{dom\sigma})$. Since μ is idempotent, μ_1 and μ_2 are idempotent too. If $y \in range\delta$ then $y \notin dom\delta$ and therefore $y\mu = y\mu_1$. So, for all $x \in dom\delta$ we have $x\delta\mu = x\delta\mu_1$ and for all $x \notin dom\delta$: $x\delta\mu = x\mu = x\mu_1 = x\delta\mu_1$ together: $\delta\mu = \delta\mu_1$. Analogously we can show that $\sigma\mu = \sigma\mu_2$.
Since the reduced weak unifier was arbitrary chosen, for any upper bound $\delta\delta' = \sigma\sigma'$ we have $\quad \delta\mu_1\psi = \delta\mu\psi = \delta(\delta' + \sigma') = \delta\delta' = \sigma\sigma' = \sigma(\delta' + \sigma') = \sigma\mu\psi = \sigma\mu_2\psi$.
and $\quad dom\mu_1 \cup dom\mu_2 \subseteq dom\mu \subseteq VAR(\delta, \sigma)$.
This concludes the proof. □

At last in this section we introduce a special substitution $\Delta_{t,u}$ which will be used in order to define L-narrowing.
If t is any term, $u \in focc(t)$, and $n = ary(t(u))$ then $\Delta_{t,u}$ is the substitution with

$$dom\Delta_{t,u} = X_n \text{ and } x_i\Delta_{t,u} := t/ui \text{ for } 1 \leq i \leq n.$$

3 L-narrowing

As already mentioned in the introduction we are dealing with rules of the form $P \Rightarrow \underline{x}f \to r$ where P is an equation system from $ES(F \cup X_n \cup X_s)$, $\underline{x}$ is a list $x_1 \ldots x_n$ of positional variables, f is a n-ary function symbol, and r is a term from $\mathcal{T}(F \cup X_n \cup X_s)$. Because by such a rule the symbol f gets a certain meaning it is called an $f-rule$. However, in this paper we are not interested in the meaning of the function symbols defined by rules, i.e. the models. Our aim is only to study the behaviour of the rules. Nevertheless, the purpose of these rules is to have a derivation mechanism in order solve equations. An equation system, also called *goal*, is represented by a term $S \in ES$.This goal is solved, if S can be derived to the empty system Λ by means of the available rules.

This used denotation is not very legible. However, it is well suitable for theoretical reasons. Moreover, we shorten this denotation even more since for an $f-rule$ the part $\underline{x}f$ is always the same.

Definition 3.1 (Rules) *A pair (P, r) where $P \in ES(F \cup X_n \cup X_s)$ and $r \in \mathcal{T}(F \cup X_n \cup X_s)$ is called a* rule *of arity n. P is called the* premise *and r is called the* result*. If a denotes a rule then by P_a we mean the premise of a and by r_a we denote the result of a, i.e. $a = (P_a, r_a)$. An $F-rule\,system$ $R_F = (R_f)_{f \in F}$ is an F indexed family of finite sets of rules where R_f contains rules of arity ary(f). Each element of R_f is called an $f-rule$.*

Next, we go on to define the application of rules. In principle, an f-rule $a = (P_a, r_a)$ can be applied at occurrence u of a goal S if

(i) the term $\underline{x}f$ matches with S/u and
(ii) the premise P_a holds after the match.

Due to our special kind of rules, the condition (i) is fulfilled if $S(u) = f$. The match is then described by the substitution $\Delta_{S,u}$. But in order to check whether the premise holds after the match we have to derive the modified P_a to Λ. This modification is done by $P_a\Delta_{S,u}$. We postpone the check and apply the f-rule a onto S at u if $S(u) = f$ in the hope that $P_a\Delta_{S,u}$ holds. The application of a onto S at u replaces in S the subterm S/u by the result of rule a regarding the match. Additionally, in order to avoid conflicts of variables, we rename in r_a each standard variable x of rule a by the auxillary variable $x^{(k)}$ using the substitution $\rho^{(k)}$ before the replacement is carried out. The substitution $\rho^{(k)}$ can be formally defined by $dom\rho^{(k)} = X_s$ and $\forall x \in X_s \; : \; x\rho^{(k)} = x^{(k)}$ where k is a certain natural number. So, we have $r_a\rho^{(k)}\Delta_{S,u}$ as the new subterm at u and the rewritten goal is $S[u \leftarrow r_a\rho^{(k)}\Delta_{S,u}]$. The join of the rewritten goal with the modified premise $P_a\rho^{(k)}\Delta_{S,u}$ implements the postponement of the premise check. The new goal is now in detail

$$S[u \leftarrow r_a\rho^{(k)}\Delta_{S,u}] \bowtie (P_a\rho^{(k)}\Delta_{S,u}).$$

Since S does not contain any positional variables and u is a position in S we can shift the substitution $\Delta_{t,u}$ to the right and we get

$$S[u \leftarrow r_a\rho^{(k)}\Delta_{S,u}] \bowtie (P_a\rho^{(k)}\Delta_{S,u}) \;=\; (S \bowtie P_a\rho^{(k)})[u \leftarrow r_a\rho^{(k)}]\Delta_{S,u}$$

This application of f-rule a at u of S is called a *L-narrowing* step. It should bring a goal closer to the application of a *reflecting* step, where an equation e within the goal is removed by finding a most general unifier of e. If goal S has length n, $1 \le i \le n$, and $\mu = mgu(S/i)$ then by a reflecting step at i we get the new goal

$$S\mu[i \leftarrow \epsilon] \;=\; S[i \leftarrow \epsilon]\mu.$$

The $mgu\,\mu$ is something important because, may be, it describes a part of the solution of the goal. All most general unifiers used in reflecting steps are therefore composed.

After this preparation we are in the position to define the derivations.

Definition 3.2 (Derivations) *Let S be an equation system of length n, called* goal*, δ an idempotent substitution, and k a natural number.*
Reflecting step: *If $1 \le i \le n$ and $\mu = mgu(S/i)$ then*

$$\langle k : S, \delta\rangle \stackrel{(i)}{\longmapsto} \langle k : S[i \leftarrow \epsilon]\mu, \delta\mu\rangle$$

L-narrowing step: *If* $u \in focc(S)$, $a = (P_a, r_a)$ *is a* $S(u) - rule$ *in* $R_{S(u)}$, $\hat{P}_a = P_a \rho^{(k)}$, *and* $\hat{r}_a = r_a \rho^{(k)}$ *then*

$$\langle k : S, \delta \rangle \overset{(u:a)}{\longmapsto} \langle k+1 : (S \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S,u}, \delta \rangle.$$

If α *is any sequence consisting of elements* (i) *or* $(u : a)$ *then we define the derivation*

$$\langle k : S, \delta \rangle \overset{\alpha}{\longmapsto} \langle l : Q, \sigma \rangle$$

by

$$\langle k : S, \delta \rangle \overset{\epsilon}{\longmapsto} \langle k : S, \delta \rangle \qquad \textit{and}$$

$$\langle k : S, \delta \rangle \overset{s\alpha}{\longmapsto} \langle l : Q, \sigma \rangle \; iff \; \langle k : S, \delta \rangle \overset{s}{\longmapsto} \langle k' : S', \delta' \rangle \overset{\alpha}{\longmapsto} \langle l : Q, \sigma \rangle.$$

where s *is either an element* (i) *or* $(u : a)$.
The sequence α *is called* derivation history.

Unfortunately, these notations are overloaded with technical overhead. But we can improve the legibility dropping some parts which can be understood by the context. Firstly, we omit the number k. This number is only used in order to avoid conflicts of variables. We take now always into consideration that by a L-narrowing step only *fresh* auxillary variables are introduced. This is done changing the premise P_a and the result r_a of rule a into $\hat{P}_a$ and $\hat{r}_a$. Secondly, we also omit the indices of the standard substitution Δ since they are mentioned before. In the abbreviated version we have for the reflecting step:

$$\langle S, \delta \rangle \overset{(i)}{\longmapsto} \langle S[i \leftarrow \epsilon]\mu, \delta\mu \rangle$$

and for the L-narrowing step:

$$\langle S, \delta \rangle \overset{(u:a)}{\longmapsto} \langle (S \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta, \delta \rangle.$$

Finally, if we are not interested in the derivation history we write simply

$$\langle S, \delta \rangle \longmapsto \langle Q, \sigma \rangle \;\; \text{instead of} \;\; \langle S, \delta \rangle \overset{\alpha}{\longmapsto} \langle Q, \sigma \rangle.$$

The following propositions are easily to prove but, nevertheless, very important for further considerations.

Proposition 3.3 $\langle S, \varphi \rangle \overset{\alpha}{\longmapsto} \langle Q, \sigma \rangle \;\Rightarrow\; \exists \psi : \; \sigma = \varphi\psi$

Proof: (By induction of the length of α.)
If $\alpha = \epsilon$ then $\sigma = \varphi$ and $\psi = \lambda$.
If $\alpha = s\alpha'$ then we have $\quad \langle S, \varphi \rangle \overset{s}{\longmapsto} \langle S', \varphi' \rangle \overset{\alpha'}{\longmapsto} \langle Q, \sigma \rangle$
and by induction: $\sigma = \varphi'\psi$. Because s is either a reflecting step or a L-narrowing step it holds $\varphi' = \varphi\mu$ or $\varphi' = \varphi$, respectively. □

Proposition 3.4 $\langle S, \varphi \rangle \overset{\alpha}{\longmapsto} \langle Q, \sigma \rangle \;\Leftrightarrow\; \langle S, \lambda \rangle \overset{\alpha}{\longmapsto} \langle Q, \psi \rangle$ *and* $\sigma = \varphi\psi$.

Proof: (By induction of the length of α.)
If $\alpha = \epsilon$ then $\psi = \lambda$, $\varphi = \sigma$, and the result follows immediately.
If $\alpha = (i)\alpha'$ then: $\quad \langle S, \varphi \rangle \overset{\alpha}{\longmapsto} \langle Q, \sigma \rangle \;\Leftrightarrow\; \langle S, \varphi \rangle \overset{(i)}{\longmapsto} \langle S', \varphi\mu \rangle \overset{\alpha'}{\longmapsto} \langle Q, \sigma \rangle$.
Now, we consider both parts individually.
By definition 3.2: $\quad \langle S, \varphi \rangle \overset{(i)}{\longmapsto} \langle S', \varphi\mu \rangle \;\Leftrightarrow\; \langle S, \lambda \rangle \overset{(i)}{\longmapsto} \langle S', \mu \rangle$.
By induction assumption:

$$\langle S', \varphi\mu \rangle \overset{\alpha'}{\longmapsto} \langle Q, \sigma \rangle \;\Leftrightarrow\; \langle S', \lambda \rangle \overset{\alpha'}{\longmapsto} \langle Q, \psi \rangle \text{ and } \sigma = \varphi\mu\psi.$$

Again by induction assumption:

$$\langle S', \lambda \rangle \overset{\alpha'}{\longmapsto} \langle Q, \psi \rangle \;\Leftrightarrow\; \langle S', \mu \rangle \overset{\alpha'}{\longmapsto} \langle Q, \sigma' \rangle \text{ and } \sigma' = \mu\psi.$$

Taking both parts together the proof is completed in the case of a reflexing step.
If $\alpha = (u : a)\alpha'$ then the proof runs analogously. □

Proposition 3.5 $\langle S, \lambda \rangle \overset{\alpha}{\longmapsto} \langle Q, \sigma \rangle \;\Rightarrow\; (dom\sigma \cup range\sigma) \subseteq (var(S) \cup X^{()})$

Proof: (By induction of the length of α.)
If $\alpha = \epsilon$ then $\sigma = \lambda$ and the proof is obvious.
If $\alpha = s\alpha'$ then we have $\langle S, \lambda \rangle \overset{s}{\longmapsto} \langle S', \mu \rangle \overset{\alpha'}{\longmapsto} \langle Q, \sigma \rangle$ and by proposition 3.4: $\langle S', \lambda \rangle \overset{\alpha'}{\longmapsto} \langle Q, \psi \rangle$ and $\sigma = \mu\psi$.
By induction assumption: $dom\psi \cup range\psi \subseteq var(S') \cup X^{()}$.
If s is a reflecting step then μ is a mgu and $dom\mu \cup range\mu \subseteq var(S), var(S') \subseteq var(S)$ holds. If s is a L-narrowing step then $\mu = \lambda$ and $var(S') \subseteq var(S) \cup X^{()}$ holds. In both cases we get: $dom\mu \cup range\mu \cup dom\psi \cup range\psi \subseteq var(S) \cup X^{()}$.
Since $dom\sigma \cup range\sigma \subseteq dom\mu \cup range\mu \cup dom\psi \cup range\psi$ the proof can be finished. □

Proposition 3.6 $\langle S, \lambda \rangle \overset{\alpha}{\longmapsto} \langle Q, \sigma \rangle \;\Rightarrow\;$ *σ is idempotent*

Proof: (By induction of the length of α.)
If $\alpha = \epsilon$ then $\sigma = \lambda$ and therefore idempotent.
If $\alpha = s\alpha'$ then we have $\langle S, \lambda \rangle \overset{s}{\longmapsto} \langle S', \mu \rangle \overset{\alpha'}{\longmapsto} \langle Q, \sigma \rangle$ and by proposition 3.4: $\langle S', \lambda \rangle \overset{\alpha'}{\longmapsto} \langle Q, \psi \rangle$ and $\sigma = \mu\psi$.
ψ is idempotent by induction assumption and $var(S') \cap dom\mu = \emptyset$ holds for $\mu = mgu(S/i)$ as well as for $\mu = \lambda$.
Let us now take any $x \in range\sigma$.
If $x \in dom\mu$ then $x \in var(S)$, $x \notin range\mu$ (since μ is idempotent), therefore $x \in range\psi$ and $x \notin var(S')$. $x \in range\psi$ implies (by proposition 3.5): $x \in X^{()}$, but this contradicts to $x \in var(S)$.
It remains $x \notin dom\mu$. If now $x \in dom\sigma$ then $x \in dom\psi$. But ψ is idempotent by induction assumption and therefore, $x \notin range\psi$. This is only possible if $x \in range\mu$ (we took x from $range\sigma$!). But $x \in dom\psi$ and since ψ is idempotent: $x \notin range(\mu\psi) = range\sigma$, a contradiction too. That means, σ is idempotent! □

We mentioned already that our mechanism should allow to find solutions of equations. Therefore, we have to explain what a solution of a goal is. Usually, in the literature, this is done on the basis of models. There, a solution of an equation $st \asymp$ is a substitution φ such that $s\varphi$ coincides with $t\varphi$, if both of the terms are evaluated for any interpretation and instantiation of the variables, i.e. the equation $s\varphi = t\varphi$ is valid. φ is called a solution of goal S if all the equations of $S\varphi$ are valid. In [Pa88] (Lemma 7.7.3), it is proved that if $S\varphi$ can be reduced to the empty goal Λ by *goal reduction* then φ is a solution of S. Moreover, narrowing is sound wrt. goal reduction (Lemma 8.2.3). This stimulates us to define a solution on the basis of narrowing by the following definition.

Definition 3.7 *If* $\langle S, \lambda \rangle \overset{\alpha}{\longmapsto} \langle \Lambda, \sigma \rangle$ *then σ is called an $\underline{S-\alpha-solution}$.
We say that S holds, if for each $S-\alpha-solution$ σ there is a substitution $\tilde{\sigma}$ such that $\sigma \sim \tilde{\sigma}$ and $\tilde{\sigma} \downarrow var(S) = \lambda$.*

May be, the reader expects that if S holds then for each S-α-solution σ we have already $\sigma \downarrow var(S) = \lambda$. But, due to the auxillary variables, this is not fulfilled in general. Let us consider the following example where $F = \{c, f\}, ary(c) = 0, ary(f) = 1$, and the only $f-rule$ is $r = (x_1 y \asymp \Lambda, c)$ (or in a more convenient form: $x_1 = y \Rightarrow f(x_1) \rightarrow c$). Let us start with the goal $xfc \asymp \Lambda$ ($f(x) = c$). We explain the derivations in our formal description as well as in an informal way. At first, we can only carry out a L-narrowing step: $\langle xfc \asymp \Lambda, \lambda\rangle \stackrel{(11:r)}{\longmapsto} \langle cc \asymp xy^{(1)} \asymp \Lambda, \lambda\rangle$

By this step the subterm $f(x)$ of the goal was replaced by c in accordance of our rule and the modified premise $x = y^{(1)}$ was added to the goal. The new goal is now $\{c = c, x = y^{(1)}\}$. Next, we can carry out two reflecting steps where the first one has λ as most general unifier of equation $c = c$. It remains the goal $x = y^{(1)}$. There are two possible most general unifiers: μ and μ' where $x\mu = y^{(1)}$ and $y^{(1)}\mu' = x$. That means either we have the derivation

$$\langle cc \asymp xy^{(1)} \asymp \Lambda, \lambda\rangle \stackrel{(1)}{\longmapsto} \langle xy^{(1)} \asymp \Lambda, \lambda\rangle \stackrel{(1)}{\longmapsto} \langle \Lambda, \mu\rangle \quad \text{or}$$
$$\langle cc \asymp xy^{(1)} \asymp \Lambda, \lambda\rangle \stackrel{(1)}{\longmapsto} \langle xy^{(1)} \asymp \Lambda, \lambda\rangle \stackrel{(1)}{\longmapsto} \langle \Lambda, \mu'\rangle.$$

But, $\mu \downarrow \{x\} \neq \lambda$!

4 Forward lifting

Let us assume that there is any derivation $\langle S, \lambda\rangle \stackrel{\alpha}{\longmapsto} \langle Q, \sigma\rangle$. What can we say about derivations with history α starting with the modified goal $S\delta$?

In order to investigate this problem we restrict the substitution δ. The first reason is to avoid conflicts of variables. Therefore, we demand that δ does not use any auxillary variables, i.e. $(dom\delta \cup range\delta) \cap X^{()} = \emptyset$. Next, δ has to be idempotent. May be, this precondition can be dropped. However, it is no serious practical restriction but it facilitates the proofs.

The third reason for the restriction is a substantial one: δ has to be consistent with σ. Otherwise, we can not expect that a derivation with history α starting with $S\delta$ exists.

To shorten the proof of the lifting lemma we show some propositions. In all these propositions we assume to have the following situation:

S.1 S is a goal of length n.

S.2 δ and σ are idempotent consistent substitutions and $(dom\delta \cup range\delta) \cap X^{()} = \emptyset$.

S.3 $(\delta', \sigma') = wu(\delta, \sigma)$.

S.4 $\mu = mgu(S/i), 1 \leq i \leq n$, μ is therefore idempotent.

S.5 $\sigma = \mu\psi$, ψ is idempotent, $dom\mu \cap dom\psi = \emptyset$, and $dom\mu \cap range\psi = \emptyset$.

S.6 $\chi = mgu((S/i)\delta)$, χ is therefore idempotent.

S.7 $\delta' = \chi\tilde{\delta}$ and $dom\chi \cap dom\tilde{\delta} = \emptyset$.

S.8 $\mu\varphi = \delta\chi$ and $dom\varphi \cap dom\mu = \emptyset$.

Proposition 4.1 $dom\mu \cup dom\psi \subseteq dom\sigma$

Proof: If $x \notin dom\sigma$ then $x\mu\psi \overset{S.5}{=} x\sigma = x$. That means there is a variable y such that $x\mu = y$ and $y\psi = x$.
If $x = y$ then we have immediately $x \notin dom\mu$ and $x \notin dom\psi$ since $x\mu = x$ and $x\psi = x$.
If $x \neq y$ then $x \in range\psi$ and by S.5 $x \notin dom\mu$, $x \notin dom\psi$ follows. □

Proposition 4.2 *$range\mu \subseteq range\sigma \cup dom\sigma$*

Proof: If $x \in range\mu$ then there exists a variable y such that $x \in var(y\mu)$ and because μ is idempotent (by S.4):$x \neq y$.
If $x \notin dom\psi$ then $x \in var(y\mu\psi) \overset{S.5}{=} var(y\sigma)$. But, this means: $x \in range\sigma$.
If $x \in dom\psi$ then by S.5: $x \notin dom\mu$ and since now $x\mu\psi = x\psi$ we get $x \in dom\mu\psi \overset{S.5}{=} dom\sigma$. □

Proposition 4.3 *$(range\chi \cup dom\chi)\backslash range\delta \subseteq (range\mu \cup dom\mu)$*

Proof: We abbreviate: $e := S/i$ and $\bar{j} := 3 - j$ for $j \in \{1, 2\}$.
If $x \in dom\chi\backslash range\delta$ then $x \in var(e)$ and there must be a suitable occurrence ju such that $e(ju) = x$ and $e\delta(\bar{j}u) \neq x$. The existence of an occurrence $\bar{j}v$ follows with $v \sqsubseteq u$, $e(\bar{j}v) = y \in X$ and $x \neq y$. Since now $e/jv \neq e/\bar{j}v$ we have either $x \in dom\mu$ or $x \in range\mu$.
If $x \in range\chi\backslash range\delta$ then $x \in var(e)$ and there must be an occurrence ju such that $x \neq e\delta(ju) \in dom\chi$ and $x \in var(e(\bar{j}u))$. That means there is an occurrence v with $v \sqsubseteq u$ and $e(jv) \in X$. Since also here $e/jv \neq e/\bar{j}v$ we have either $x \in dom\mu$ or $x \in range\mu$. □

Proposition 4.4 *φ is idempotent.*

Proof: Let us take any $x \in dom\varphi$, then by S.8 $x \notin dom\mu$ and $x \in dom\mu\varphi = dom\delta\chi$.
If $x \in dom\delta$ then by S.2: $x \notin range\delta$ and $x \notin var(S\delta)$ follows. Since $\chi = mgu((S/i)\delta)$: $x \notin range\chi$ and consequently $x \notin range\delta\chi = range\mu\varphi$.
If $x \notin dom\delta$ then $x \in dom\chi$ and therefore by S.6: $x \notin range\delta\chi = range\mu\varphi$.
If $x \in range\varphi$ then there exists an y and $x \in var(y\varphi)$, i.e. $y \in dom\varphi$. But, by S.8 we have $y \notin dom\mu$ and $x \in var(y\varphi) = var(y\mu\varphi)$ follows, i.e. $x \in range\mu\varphi$, a contradiction.
This concludes the proof. □

Proposition 4.5 *If $\sigma'' := \sigma' + (\tilde{\delta} \downarrow dom\mu)$ then $\mu\psi\sigma'' = \mu\psi\sigma'$ and $\psi\sigma'' = \varphi\tilde{\delta}$.*

Proof: If $x \in dom\mu$ then by proposition 4.1: $x \in dom\sigma$ and by S.3: $x \notin dom\sigma'$. Therefore, σ'' is well defined.
If $y \in range(\mu\psi) \subseteq range\mu \cup range\psi$ then by S.4 and S.5 $y \notin dom\mu$ follows and we have $y\sigma'' = y\sigma'$. Therefore, $x\mu\psi\sigma'' = x\mu\psi\sigma'$ follows if $x \in dom(\mu\psi)$. For $x \notin dom(\mu\psi) = dom\sigma$ we get by proposition 4.1 $x \notin dom\mu \cup dom\psi$, that means $x\mu\psi\sigma'' = x\sigma'' = x\sigma' = x\mu\psi\sigma'$, altogether:

$$\mu\psi\sigma'' = \mu\psi\sigma'. \tag{1}$$

Now, we show: $\psi\sigma'' = \varphi\tilde{\delta}$.
If $x \in dom\mu$ then by S.5:

$$x \notin dom\psi \tag{2}$$

and by S.8:

$$x \notin dom\varphi \tag{3}$$

We get: $x\psi\sigma'' \overset{(2)}{=} x\sigma'' = x\tilde{\delta} \overset{(3)}{=} x\varphi\tilde{\delta}$.
If $x \notin dom\mu$ then we get:

$$x\psi\sigma'' = x\mu\psi\sigma'' \overset{(1)}{=} x\mu\psi\sigma' \overset{S.5}{=} x\sigma\sigma' \overset{S.3}{=} x\delta\delta' \overset{S.7}{=} x\delta\chi\tilde{\delta} \overset{S.8}{=} x\mu\phi\tilde{\delta} = x\varphi\tilde{\delta}.$$

This concludes the proof. □

Proposition 4.6 *If* $(\hat{\delta}, \hat{\sigma}) = wu(\varphi, \psi)$ *then* $(\chi\hat{\delta}, \hat{\sigma}\downarrow\overline{dom\sigma}) = wu(\delta, \sigma)$

Proof:
a) We show: $\delta\chi\hat{\delta} = \sigma(\hat{\sigma} \downarrow \overline{dom\sigma})$.
We get $\psi\hat{\sigma} = \varphi\hat{\delta}$ by $(\hat{\delta}, \hat{\sigma}) = wu(\varphi, \psi)$ and therefore: $\mu\psi\hat{\sigma} = \mu\varphi\hat{\delta}$. (a1)
Since by S.2 σ is idempotent we have: $\sigma\hat{\sigma} = \sigma(\hat{\sigma} \downarrow \overline{dom\sigma})$ (a2)
and we can conclude: $delta\chi\hat{\delta} \stackrel{S.8}{=} \mu\varphi\hat{\delta} \stackrel{(a1)}{=} \mu\psi\hat{\sigma} \stackrel{S.5}{=} \sigma\hat{\sigma} \stackrel{(a2)}{=} \sigma(\hat{\sigma} \downarrow \overline{dom\sigma})$.
b) We show: $\delta\chi\hat{\delta} \sqsubseteq \delta\delta'$.
By proposition 4.5 $(\psi\sigma'' = \varphi\tilde{\delta})$ and since $(\hat{\delta}, \hat{\sigma}) = wu(\varphi, \psi)$ we have $\psi\hat{\sigma} \sqsubseteq \psi\sigma''$ and therefore: $\mu\psi\hat{\sigma} \sqsubseteq \mu\psi\sigma'' = \mu\psi\sigma'$. (b1)
It follows: $\delta\chi\hat{\delta} \stackrel{a)}{=} \sigma\hat{\sigma} \stackrel{S.5}{=} \mu\psi\hat{\sigma} \stackrel{(b1)}{\sqsubseteq} \mu\psi\sigma' \stackrel{S.5}{=} \sigma\sigma' \stackrel{S.3}{=} \delta\delta'$.
c) $dom\sigma \cap dom(\hat{\sigma} \downarrow \overline{dom\sigma}) = \emptyset$ holds trivially.
d) We show: $dom\chi\hat{\delta} \cap dom\delta = \emptyset$
We take any $x \in dom\chi\hat{\delta}$ and therefore: $x \in dom\chi \cup dom\hat{\delta}$. (d1)
Since $(\hat{\delta}, \hat{\sigma}) = wu(\varphi, \psi)$: $dom\hat{\delta} \cap dom\varphi = \emptyset$. (d2)
Case 1: $x \in dom\chi$
Because of $\chi = mgu((S/i)\delta)$ and δ is idempotent we get $x \notin dom\delta$.
Case 2: $x \notin dom\chi$
Because of (d1) we have $x \in dom\hat{\delta}$. By (d2) we conclude: $x \notin dom\varphi$ (d3)
and by $(\hat{\delta}, \hat{\sigma}) = wu(\varphi, \psi)$ we get $x \in VAR(\varphi, \psi)$.
Therefore, there exists an y such that $y \in dom\varphi \cup dom\psi$ and $x \in var(y\varphi) \cup var(y\psi)$.
Now, we assume $x \in dom\delta$ and therefore, $x \notin var((S/i)\delta)$ and we have the following chain of conclusions: $(\chi = mgu(S/i)\delta)) \Longrightarrow x \notin range\chi \Longrightarrow x\delta\chi \neq x \Longrightarrow x \in dom\delta\chi \stackrel{S.8}{=} dom\mu\varphi \subseteq dom\mu \cup dom\varphi \stackrel{(d3)}{\Longrightarrow} x \in dom\mu \stackrel{(S.5)}{\Longrightarrow} x \notin range\psi \cup dom\psi$. (d4)
If $x \in range\varphi$ then there is a $z \in dom\varphi$ different from x such that $x \in var(z\varphi)$. By S.8 now $z \notin dom\mu$ follows and therefore, $x \in var(z\mu\varphi) \stackrel{S.8}{=} var(z\delta\chi)$ and $x \in range\delta\chi$ follows. But, this contradicts to our assumption $x \in dom\delta$ since δ is idempotent and $\chi = mgu((S/i)\delta)$. It remains $x \notin range\varphi$ and by (d4) we get $x = y$ and consequently, $x \in dom\varphi \cup dom\psi$. This is also a contradiction to (d3) and (d4).
e) We show: $dom\chi\hat{\delta} \cup dom(\hat{\sigma} \downarrow \overline{dom\sigma}) \subseteq VAR(\delta, \sigma)$.
We assume, that $x \notin range\delta \cup range\sigma$ (otherwise $x \in range\delta \cup range\sigma \subseteq VAR(\delta, \sigma)$ would hold and we are done).
Case 1: $x \in dom\chi\hat{\delta}$
By d) we get $x \notin dom\delta$, i.e.: $x\delta = x$. (e1)
If $x \in dom\sigma$ then we get immediately $x \in VAR(\delta, \sigma)$. Therefore, we assume $x \notin dom\sigma$.
It follows $x\chi\hat{\delta} \stackrel{(e1)}{=} x\delta\chi\hat{\delta} \stackrel{a)}{=} x\sigma\hat{\sigma} = x\hat{\sigma}$. This implies $x \in dom\hat{\sigma}$ and furthermore, since $(\hat{\delta}, \hat{\sigma}) = wu(\varphi, \psi)$: $x \in VAR(\varphi, \psi)$.
Therefore, there exists an y such that $y \in dom\varphi \cup dom\psi$ and $x \in var(y\varphi) \cup var(y\psi)$. By S.8 and S.5 we get: $y \notin dom\mu$. (e2)
Case 1.1: $x \in var(y\psi) \stackrel{(e2)}{=} var(y\mu\psi) \stackrel{S.5}{=} var(y\sigma)$.
If $x \neq y$ then $y \in dom\sigma$ and therefore, $x \in VAR(\delta, \sigma)$.
If $x = y$ then $x \notin dom\psi$ (ψ is idempotent!) and $x = y \in dom\varphi$ follows. By (e2) we get $x \in dom\mu\varphi \stackrel{S.8}{=} dom\delta\chi$. Since (e1) we have $x \in dom\chi$. Now, we can apply proposition

4.3 and get $x \in range\mu$. By proposition 4.2 we have $x \in range\sigma \subseteq VAR(\varphi, \psi)$. This concludes case 1.1.
Case 1.2: $x \in var(y\varphi)$, then by (e2): $x \in var(y\mu\varphi) \stackrel{S.8}{=} var(y\delta\chi)$.
Case 1.2.1: $x \in var(y\delta)$.
If $x \neq y$ then $x \in range\delta \subseteq VAR(\varphi, \psi)$.
If $x = y$ then $x \notin dom\varphi$ and $x \notin dom\delta$ and by (e2): $x \notin dom\mu$. By $y = x \notin dom\varphi$ we get $x = y \in dom\psi$ and therefore, $x\psi = x\mu\psi \stackrel{S.5}{=} x\sigma$ and $x \in dom\sigma$ follows, i.e. $x \in VAR(\varphi, \psi)$.
Case 1.2.2: $x \notin var(y\delta)$.
Because of $x \notin range\delta$ there is a variable $z \in var(y\delta)$ such that $x \in var(z\chi)$ and $x \neq z$. Now:

$$z \in dom\chi \text{ and } x \in range\chi. \qquad \text{(e3)}$$

If $x \in dom\mu$ then by proposition 4.1 we get $x \in dom\sigma$, but this contradicts to our assumption that $x \notin dom\sigma$.
It remains $x \notin dom\mu$. (e2) and (e3) allows to apply proposition 4.3 and we get $x \in range\mu$. The proposition 4.2 tells us, that $x \in range\sigma \subseteq VAR(\varphi, \psi)$, which concludes this case.
Case 2: $x \in dom(\hat{\sigma} \downarrow \overline{dom\sigma})$.
This case runs analogously to case 1. □

Lemma 4.7 (Forward lifting) *If* $\langle S, \lambda \rangle \stackrel{\alpha}{\longmapsto} \langle Q, \sigma \rangle$ *and* δ *is an idempotent substitution consistent with* σ *and* $(dom\delta \cup range\delta) \cap X^{()} = \emptyset$ *then there exists a derivation*

$$\langle S\delta, \lambda \rangle \stackrel{\alpha}{\longmapsto} \langle Q\sigma', \delta' \rangle \qquad \textit{where} \qquad (\delta', \sigma') = wu(\delta, \sigma).$$

Proof: (By induction on the length of α.)
The preconditions of this lemma guarantee that S.1, S.2, and S.3 hold. If $\alpha = \epsilon$ then

$$\langle S\delta, \lambda \rangle \stackrel{\epsilon}{\longmapsto} \langle S\delta, \lambda \rangle$$

and $\quad (\lambda, \delta) = wu(\delta, \lambda)$.
If $\alpha = (i)\alpha'$ then

$$\langle S, \alpha \rangle \stackrel{(i)}{\longmapsto} \langle S[i \leftarrow \epsilon]\mu, \mu \rangle \stackrel{(\alpha')}{\longmapsto} \langle Q, \sigma \rangle$$

and $\quad \mu = mgu(S/i)$ and by proposition 3.4:

$$\langle S[i \leftarrow \epsilon]\mu, \lambda \rangle \stackrel{(\alpha')}{\longmapsto} \langle Q, \psi \rangle, \text{ where } \sigma = \mu\psi.$$

Now, S.4 holds because of the assumption: $\alpha = (i)\alpha'$, and S.5 holds because of propositions 3.5 and 3.6.

Since μ is an unifier of S/i it is $\mu\psi\sigma'$ an unifier of S/i and therefore $\delta\delta' = \sigma\sigma' = \mu\psi\sigma'$ is an unifier of S/i too. That means: δ' is an unifier of $(S/i)\delta$. Now, the existence of $\chi := mgu((S/i)\delta)$ follows, i.e. S.6 holds.

Since δ' is an unifier of $(S/i)\delta$ and $\chi = mgu((S/i)\delta)$ there exists a substitution $\tilde{\delta}$ with $\chi\tilde{\delta} = \delta'$ amd $dom\chi \cap dom\tilde{\delta} = \emptyset$. This is S.7.

Finally, S.8 is fulfilled since $\mu = mgu(S/i)$ and $\delta\chi$ is an unifier of S/i and we can conclude the existence of a substitution φ with $\mu\varphi = \delta\chi$ and $dom\varphi \cap dom\mu = \emptyset$. By proposition 4.4 we know that φ is idempotent and by proposition 4.5 that φ and ψ are consistent where ψ is also idempotent by proposition 3.6. According to the induction the derivation

$$\langle S[i \leftarrow \epsilon]\mu, \lambda \rangle \stackrel{(\alpha')}{\longmapsto} \langle Q, \psi \rangle$$

implies a derivation

$$\langle S[i \leftarrow \epsilon]\mu\varphi, \lambda \rangle \stackrel{(\alpha')}{\longmapsto} \langle Q\hat{\sigma}, \hat{\delta} \rangle$$

where $(\hat{\delta}, \hat{\sigma}) := wu(\varphi, \psi)$ (it exists because of lemma 2.6). Since $\mu\varphi = \delta\chi$ we have also a derivation

$$\langle S[i \leftarrow \epsilon]\delta\chi, \lambda\rangle \overset{(\alpha')}{\longmapsto} \langle Q\hat{\sigma}, \hat{\delta}\rangle.$$

The application of proposition 3.4 leads to a derivation

$$\langle S[i \leftarrow \epsilon]\delta\chi, \chi\rangle \overset{(\alpha')}{\longmapsto} \langle Q\hat{\sigma}, \chi\hat{\delta}\rangle.$$

According to definition 3.2 we get as the first step:

$$\langle S\delta, \lambda\rangle \overset{(i)}{\longmapsto} \langle S[i \leftarrow \epsilon]\delta\chi, \chi\rangle$$

and together:

$$\langle S\delta, \lambda\rangle \overset{\alpha}{\longmapsto} \langle Q\hat{\sigma}, \chi\hat{\delta}\rangle.$$

But, $var(Q) \subseteq \overline{dom\sigma}$ and that means $Q\hat{\sigma} = Q(\hat{\sigma} \downarrow \overline{dom\sigma})$ and the existence of a derivation

$$\langle S\delta, \lambda\rangle \overset{\alpha}{\longmapsto} \langle Q(\hat{\sigma} \downarrow \overline{dom\sigma}), \chi\hat{\delta}\rangle.$$

follows, where $(\chi\hat{\delta}, \hat{\sigma} \downarrow \overline{dom\sigma}) = wu(\delta, \sigma)$ (by proposition 4.6).
If $\alpha = (u : a)\alpha'$ then

$$\langle S, \lambda\rangle \overset{(u:a)}{\longmapsto} \langle (S \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S,u}, \lambda\rangle \overset{(\alpha')}{\longmapsto} \langle Q, \sigma\rangle.$$

By induction it immediately follows that

$$\langle (S \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S,u}\delta, \lambda\rangle \overset{(\alpha')}{\longmapsto} \langle Q\sigma', \delta'\rangle$$

where $(\delta', \sigma') = mgwu(\delta, \sigma)$. The first step is in accordance to definition 3.2:

$$\langle S\delta, \lambda\rangle \overset{(u:a)}{\longmapsto} \langle (S\delta \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S\delta,u}, \lambda\rangle.$$

But, according S.2, δ does not use any auxillary variables it is not hard to see that

$$(S\delta \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S\delta,u} = (S \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S,u}\delta.$$

This proves this case and we are done. □

A special case of this last lemma is of particular interest, namely if we apply the derived S-α-solution on S and then start the derivation again with $S\sigma$. Unfortunately, the new solution does not coincide with λ, in general. The reason consists in the fact that σ contains parts where auxillary variables are substituted in order to solve equations which were added to the original goal S during the derivation by L-narrowing steps. Applying σ on S the information about these steps are lost since S does not contain any auxillary variables. Therefore, these steps are repeated again. Moreover, if we want to apply lemma 4.7 we are not allowed to start with $\delta = \sigma$ because of the precondition $(dom\delta \cup range\delta) \cap X^{()} = \emptyset$.

For this reason we start with a restricted and modified σ. We introduce a substitution ρ with $dom\rho = range\sigma$ and $\forall x, y \in dom\rho : x\rho \in (X_s \backslash var(S))$and if $x\rho = y\rho$ then $x = y$. ρ replaces all auxillary variables of $range\sigma$ by standard variables not occurring in S.

Theorem 4.8 (Soundness) *If σ is a S–α-solution and $\delta = (\sigma\rho) \downarrow var(S)$ then $S\delta$ holds.*

Proof: Since σ is a $S-\alpha$-solution there is a derivation $\langle S, \lambda\rangle \overset{\alpha}{\longmapsto} \langle \Lambda, \sigma\rangle$. σ and δ are idempotent. We consider the substitution $\hat{\delta} := (\sigma\rho) \downarrow X^{()}$. It is $dom\delta \cap dom\hat{\delta} = range\delta \cap dom\hat{\delta} = \emptyset$ and therefore, $\delta\hat{\delta} = \delta + \hat{\delta} = \sigma\rho$. That means: δ and σ are consistent and $(\hat{\delta}, \rho_{Z,Y})$ is a weak unifier of (δ, σ).
We apply our lemma 4.7 and get a derivation

$$\langle S\delta, \lambda\rangle \overset{\alpha}{\longmapsto} \langle \Lambda\sigma', \delta'\rangle \text{ where } (\delta', \sigma') = wu(\delta, \sigma).$$

Now, there exists a substitution κ with $\delta\delta'\kappa = \delta\hat{\hat{\delta}}$.
Let us take any variable $x \in var(S\delta)$. Then $x \notin dom\delta$ and $x \notin X^{()}$ that means, $x \notin dom\hat{\hat{\delta}}$.
We get: $x\delta'\kappa = x\delta\delta'\kappa = x\delta\hat{\hat{\delta}} = x\hat{\hat{\delta}} = x$.
Moreover, let x, y be any variables of $var(S\delta)$. If $x\delta' = y\delta'$ then $x = x\delta'\kappa = y\delta'\kappa = y$.
If now $x\delta' = y$ and $x \neq y$ then $x\delta'\delta' \neq x\delta'$, that means $\delta'\delta' \neq \delta'$. But this contradicts to the idempotence of δ'. Altogether, δ' maps the variables of $var(S\delta)$ one-to-one into variables outside of $var(S\delta)$.
We construct now a renaming ϕ by:

$$x\phi = \begin{cases} x\delta' & \text{if} \quad x \in var(S\delta) \\ y & \text{if} \quad x = y\delta' \text{ and } y \in var(S\delta) \\ x & \text{otherwise} \end{cases}$$

It is $\delta' \sim \delta'\phi$ and $\delta'\phi \downarrow var(S\delta) = \lambda$, that means $S\delta$ holds! □

5 Backward lifting

We now try to get an analogous result of lemma 4.7 but in the opposite direction. We assume that there is a derivation with history α which starts with a goal $S\delta$. What can we say about a derivation of history α starting with S only?

In order to be sure that also the second derivation exists we have to care that all parts in the first derivation which are introduced by the substitution δ are not touched. This is the main problem for the backward lifting case. We overcome this problem by colouring red all goals involved in a derivation. Then we forbid that lazy narrowing steps are applied onto red occurrences. But, using induction, we must show that the second derivation also does not use steps onto red occurrences. This will be prepared next.

We allow that terms as well as substitutions have red coloured occurrences. The colouring is always downward closed, i.e. if $t(u)$ is red and $v \in occ(t)$ with $u \sqsubseteq v$ (v is *below* U) then v is also red. We say that the substitution φ is red iff $\forall x \in dom\varphi : (x\varphi)(\epsilon)$ is red. The consequence for a red substitution φ is that all terms $x\varphi$ are completely red if $x \in dom\varphi$.

If we apply a coloured substitution on a term then the red occurrences are transmitted too. The same holds for the composition of substitutions: if $(x\varphi)(u) = y \in X$ and $(y\psi)(v)$ is red then $(x\varphi\psi)(uv)$ is red too, and if $(x\varphi)(u)$ is red then $(x\varphi\psi)(uv)$ is red for all v.

If $\mu \sqsubseteq \delta$ then by $\delta\backslash\mu$ we mean the substitution φ such that $\mu\varphi = \delta$. φ exists and it is unique up to a renaming. We build the set

$$U := \{(u, x) \ : \ x \in dom\mu \cup dom\delta \text{ and } u \in vocc(x\mu)\}.$$

Since $\forall x \in X : x\mu\varphi = x\delta$, it follows $\forall (u, x) \in U : (x\mu\varphi)(u) = (x\delta)(u)$. φ is now coloured as follows:
If there is a $(u, x) \in U$ such that v is a red occurrence in $(x\delta)/u$ then v is coloured red in $(x\mu)(u)\varphi$.

If e is an equation which may contain some red occurrences and μ is a most general unifier of e. Then this unifier should inherit from e red labels too. This is done nondeterministically. In order to describe this we give firstly a very simple algorithm for the mgu.

We introduce:

$O(e) := \{ju \ : \ j \in \{1,2\} \wedge ju \in vocc(e) \wedge e(ju) \neq e(\bar{j}u)\}$ where $\bar{j} := 3 - j$.

$\underline{mgu(e):}$

$n := 0;\ e_n := e;\ \mu_n := \lambda;$

while $O(e_n) \neq \emptyset$ do begin

(1) take any $ju \in O(e_n)$;

(2) build the substitution ψ_n where:

$$x\psi_n = \begin{cases} e_n/\bar{j}u & \text{if } e(ju) = x \text{ and additionally } x\psi_n(\epsilon) \text{ becomes red if } ju \text{ is red in } e_n, \\ x & \text{otherwise.} \end{cases}$$

(3) $e_{n+1} := e_n\psi_n;\ \mu_{n+1} := \mu_n\psi_n;\ n := n+1$ end of mgu.

This algorithm terminates if the mgu exists. We are only interested in this case.

We call the occurrence $ju \in O(e_n)$ *clean* if $e_n(ju)$ and $e_n(\bar{j}u)$ both are not red.

Proposition 5.1 *If $O(e_n)$ does not have any clean elements then $O(e_{n+1})$ also does not have clean elements.*

Proof: Let us assume that $ju \in O(e_{n+1})$ is clean. Then $ju \notin O(e_n)$. Therefore, either ju or $\bar{j}u$ was introduced in step n into e_{n+1} by ψ_n. Since $O(e_n)$ does not have clean elements all $x\psi_n$ with $x \in dom\psi_n$ are completely red and ju or $\bar{j}u$ is red in e_{n+1}, i.e. ju is not clean, a contradiction. □

Proposition 5.2 *Let δ be a red substitution and e be an equation.*
If in the first n steps of $mgu(e\delta)$ only clean elements of $O(e_n)$ are taken then in the first n steps of $mgu(e)$ clean elements can be taken too.
Moreover, if e_n means the equation after n steps of $mgu(e\delta)$ and $\hat{e}_n$ means the corresponding equation of $mgu(e)$ then we have:
If $e_n(v)$ is not red then $\hat{e}_n(v)$ is not red and $e_n(v) = \hat{e}_n(v)$.

Proof: (by induction on n.)
If $n = 0$ then the proof is obvious.
Let us assume the proposition holds for n and the clean element $ju \in O(e_n)$ is taken in $mgu(e\delta)$. Then there is an x with $e_n(ju) = x$ and $x\psi_n = e_n/\bar{j}u$. By induction assumption we have: $\hat{e}_n(ju) = e_n(ju) = x$ and $\hat{e}_n(\bar{j}u) = e_n(\bar{j}u)$, that means: $ju \in O(\hat{e}_n)$ and ju is clean. This ju we use also in the next step of $mgu(e)$ and we get: $x\hat{\psi}_n = \hat{e}_n/\bar{j}u$.
Let us now consider any $v \in occ(e_{n+1})$ where $e_{n+1}(v)$ is not red. If $v \in occ(e_n)$ then the induction assumption verifies this case.
If $v \notin occ(e_n)$ then v was freshly introduced in step n using ju and there are v_1, v_2 with $v = v_1v_2, e_n(v_1) = x$, and $v_2 \in occ(x\psi_n)$. By induction assumption we know that $\hat{e}_n(v_1) = x$. From $x\psi_n = e_n/\bar{j}u$ and $v_2 \in occ(x\psi_n)$ we conclude: $\bar{j}uv_2 \in occ(e_n)$ and $e_n(\bar{j}uv_2)$ cannot be red. This implies again by induction assumption that $\hat{e}_n(\bar{j}uv_2)$ is not red. Finally, we have $v_2 \in occ(x\hat{\psi}_n)$ and therefore, $v = v_1v_2 \in occ(\hat{e}_n\hat{\psi}_n) = occ(\hat{e}_{n+1})$. □
The last both propositions together lead to:

Proposition 5.3 *If δ is red, $\chi = mgu(e\delta)$, e does not have red occurrences, and $\mu = mgu(e)$ then every non red occurrence u of $e\delta\chi$ is an occurrence of $e\mu$ and $(e\delta\chi)(u) = (e\mu)(u)$.*

Proposition 5.4 *Let δ be red and $\chi = mgu(e\delta)$. If $x \in dom\chi$ and v is a non red occurrence of $x\chi$ then $x \in var(e)$.*

Proof: Let us assume that for the mentioned variable x after n steps the first time $x \in dom\psi_n$ (such an n exists since $x \in dom\chi$). Then we have $e_n(ju) = x$ and ju is not red. But this means that $ju \in occ(e\chi_n)$ and therefore, $ju \in occ(e\psi_1 \ldots \psi_n)$. Since ju is not red, the variable x cannot be in $range\delta$, that means: $x \in var(e)$. □

Proposition 5.5 *Let be: $\chi = mgu(e\delta)$, $\mu = mgu(e), \varphi = \delta\chi\backslash\mu$, δ be red.*
If for any x the occurrence u is red in $x\mu\varphi$ then u is red in $x\delta\chi$.

Proof: We take any red u from $occ(x\mu\varphi)$. This must be coloured by φ and we have suitable y, v, v' with $u = vv' \in occ(x\mu\varphi)$, $(x\mu)(v) = y \in X$, $y \in dom\varphi$.
If $x \in dom\delta$ then u is red in $x\delta\chi$. If $x \notin dom\delta$ then we have

$$x\mu\varphi = x\delta\chi = x\chi \text{ and } y\varphi = (x\chi)/v.$$

This means $x \in dom\chi$ and consequently, $x \in var(e\delta)$ since $\chi = mgu(e\delta)$. If v would be not red in $x\chi$ then by proposition 5.4 $x \in var(e)$ follows and there is a non red occurrence w with $e(w) = x$. Now, we can already conclude that wv is not red in $e\delta\chi$.
By proposition 5.3 we know that wv is also not red in $e\mu$ and, of course, w is not red in $e\mu$ (otherwise wv would be red). Moreover, from proposition 5.3 the both equations:

$$(e\mu)(wv) = (e\delta\chi)(wv) = (e\chi)(wv) \text{ and } (e\mu)(w) = (e\delta\chi)(w) = (e\chi)(w)$$

follow. That means: $y = (x\mu)(v) = (x\chi)(v)$ and therefore, $y\varphi = y$. But this is impossible because of $y \in dom\varphi$, a contradiction! It remains: v is red in $x\chi = x\delta\chi$. □

Proposition 5.6 *If $\langle S, \lambda\rangle \overset{\alpha}{\longmapsto} \langle Q, \sigma\rangle$ and $\langle S', \lambda\rangle \overset{\alpha}{\longmapsto} \langle Q', \sigma'\rangle$, where S coincides with S' up to colours and every red occurrence of S' is a red occurrence of S and no element of α is in a red occurrence of the corresponding goal of the first derivation then no element of α is in a red occurrence of the corresponding goal in the second derivation.*

Proof: We omit the proof. It can be done by induction using the property that if S' has no more red occurrences than S then this relationship is preserved after one derivation step. □

Now, we are in a position to prove our

Lemma 5.7 (Backward lifting) *If δ is a red substitution with $(dom\delta \cup range\delta) \cap X^{()} = \emptyset$ and $\langle S\delta, \lambda\rangle \overset{\alpha}{\longmapsto} \langle Q, \delta'\rangle$ and no element of α is in a red occurrence of the corresponding goal then*

$$\langle S, \lambda\rangle \overset{\alpha}{\longmapsto} \langle Q', \sigma\rangle, \text{ where } Q = Q'\sigma' \text{ and } \delta\delta' = \sigma\sigma'.$$

Proof: (By induction on the length of α.)
If $\alpha = \epsilon$ then we have

$$\langle S\delta, \lambda\rangle \overset{\epsilon}{\longmapsto} \langle S\delta, \lambda\rangle \text{ and } \langle S, \lambda\rangle \overset{\epsilon}{\longmapsto} \langle S, \lambda\rangle \text{ and } \delta\lambda = \delta = \lambda\delta.$$

If $\alpha = (i)\alpha'$ then we have

$$\langle S\delta, \lambda\rangle \overset{(i)}{\longmapsto} \langle S[i \leftarrow \epsilon]\delta\chi, \chi\rangle \overset{\alpha'}{\longmapsto} \langle Q, \chi\psi\rangle$$

where $\delta' = \chi\psi$ and $\chi = mgu((S\delta)/i)$.
According to proposition 3.4 the last part becomes

$$\langle S[i \leftarrow \epsilon]\delta\chi, \lambda\rangle \overset{\alpha'}{\longmapsto} \langle Q, \psi\rangle.$$

$\delta\chi$ is an unifier of S/i and therefore, $\mu = mgu(S/i)$ exists and there is a φ with $\mu\varphi = \delta\chi$, i.e. $\varphi = \delta\chi\backslash\mu$. This leads to the derivation

$$\langle S[i \leftarrow \epsilon]\mu\varphi, \lambda\rangle \overset{\alpha'}{\longmapsto} \langle Q, \psi\rangle.$$

By proposition 5.5 it follows that each red occurrence of $S[i \leftarrow \epsilon]\mu\varphi$ is a red occurrence of $S[i \leftarrow \epsilon]\delta\chi$ and by proposition 5.6 no element of α' is in a red occurrence of the corresponding goal and we obtain by induction a derivation

$$\langle S[i \leftarrow \epsilon]\mu, \lambda\rangle \overset{\alpha'}{\longmapsto} \langle Q', \sigma\rangle, \text{ where } \varphi\psi = \sigma\sigma' \text{ and } Q = Q'\sigma'.$$

We use again proposition 3.4 and get:

$$\langle S[i \leftarrow \epsilon]\mu, \mu\rangle \overset{\alpha'}{\longmapsto} \langle Q', \mu\sigma\rangle.$$

Together with the first step we get:

$$\langle S, \lambda\rangle \overset{(i)}{\longmapsto} \langle S[i \leftarrow \epsilon]\mu, \mu\rangle \overset{\alpha'}{\longmapsto} \langle Q', \mu\sigma\rangle \text{ where } \delta\delta' = \delta\chi\psi = \mu\varphi\psi = \mu\sigma\sigma'.$$

If $\alpha = (u : a)\alpha'$ then we have

$$\langle S\delta, \lambda\rangle \overset{(u:a)}{\longmapsto} \langle (S\delta \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S\delta,u}, \lambda\rangle \overset{\alpha'}{\longmapsto} \langle Q, \delta'\rangle.$$

Since u is not red in $S\delta$ we have $u \in occ(S)$ and

$$\langle (S\delta \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S\delta,u}, \lambda\rangle = \langle (S \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S,u}\delta, \lambda\rangle$$

and by induction, there is the derivation:

$$\langle (S \bowtie \hat{P}_a)[u \leftarrow \hat{r}_a]\Delta_{S,u}, \lambda\rangle \overset{\alpha'}{\longmapsto} \langle Q', \sigma\rangle, \text{ where } \delta\delta' = \sigma\sigma' \text{ and } Q = Q'\sigma'.$$

Combining this together with the first step we are done. □

Like Theorem 4.8 we specialize now the preconditions of the backward lifting lemma.

Theorem 5.8 (Completeness) *If* $\langle S\delta, \lambda\rangle \overset{\alpha}{\longmapsto} \langle \Lambda, \delta'\rangle$, $dom\delta \subseteq var(S)$, $\delta' \sim \hat{\delta}$, $\hat{\delta} \downarrow var(S) = \lambda$ *(i.e. $S\delta$ holds!) and no element of α is in a red occurrrence of a goal then* $\langle S, \lambda\rangle \overset{\alpha}{\longmapsto} \langle \Lambda, \sigma\rangle$ *and there exists an $\hat{\sigma}$ such that* $(\sigma\hat{\sigma})\downarrow var(S) = \delta$.

Proof: By the precondition of the theorem, $dom\delta \subseteq var(S)$ and $S\delta$ holds, δ' is a $S-\alpha-$solution, $\delta' \sim \hat{\delta}$ and $\hat{\delta} \downarrow var(S) = \lambda$. Because of the derivation $\langle S\delta, \lambda\rangle \overset{\alpha}{\longmapsto} \langle \Lambda, \delta'\rangle$ and by lemma 5.7 a derivation $\langle S, \lambda\rangle \overset{\alpha}{\longmapsto} \langle Q', \sigma\rangle$ exists, where $\Lambda = Q'\sigma'$ and $\delta\delta' = \sigma\sigma'$. But, if $\Lambda = Q'\sigma'$ then $Q' = \Lambda$ must hold. Since $\delta' \sim \hat{\delta}$, there is an $\hat{\sigma}$ such that $\delta\hat{\delta} = \sigma\hat{\sigma}$, $\Lambda\hat{\sigma} = \Lambda$, and $(\sigma\hat{\sigma})\downarrow var(S) = (\delta\hat{\delta})\downarrow var(S) = \delta(\hat{\delta}\downarrow var(S\delta))\downarrow var(S) = \delta$. □

This theorem seems to be very weak, because the $S-\alpha$-solution δ is only described by $\sigma\sigma'$. σ is a certain beginning of the solution. Can we say more about σ?

We slightly modify σ and start with $\tilde{\delta} := (\sigma\rho) \downarrow var(S)$ again where ρ replaces all auxillary variables of $range\sigma$ by standard variables not occurring in S. $\tilde{\delta}$ fulfills the preconditions of theorem 4.8 and therefore, $S\tilde{\delta}$ holds. That means, there is a derivation $\langle S\tilde{\delta}, \lambda\rangle \overset{\alpha}{\longmapsto} \langle \Lambda, \delta'\rangle$, $\delta' \sim \hat{\delta}$, and $\hat{\delta} \downarrow var(S\tilde{\delta}) = \lambda$. This is closely related to the usual completeness results of narrowing. By L-narrowing, as it is shown by means of the example in the introduction, even reducible solutions can be derived under certain circumstances.

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